GS [125 marks]

1. [Maximum mark: 8]

Consider a sequence of ten rectangular picture frames $F_1, F_2, \ldots, F_9, F_{10}$.

Picture frame F_1 has width $4\ \text{cm}$ and height $5\ \text{cm}$.

The width and height of picture frame F_n, are each increased by 50% to generate the width and height of the next picture frame F_{n+1}, for $n \in \mathbb{Z}^+, 1 \le n \le 9$.

(a.i) Show that the area of picture frame F_n is $20\left(\frac{9}{4}\right)^{n-1}\ \text{cm}^2$. [2]

Markscheme

for a sequence of areas, uses two consecutive terms to find a common ratio **OR** for sequences of both widths and heights uses two consecutive terms for both sequences to find both common ratios **OR** recognises that both widths and heights are geometric sequences with common ratio $\frac{3}{2}$ ***M1***

areas form a geometric sequence with first term 20 and common ratio $\frac{45}{20}$ ***A1***

OR area of picture frame F_n is $4\left(\frac{3}{2}\right)^{n-1} \times 5\left(\frac{3}{2}\right)^{n-1}$

area of F_n is $20\left(\frac{9}{4}\right)^{n-1}$ ***AG***

[2 marks]

(a.ii) Hence, find the mean area of the ten picture frames, giving your answer in the form $p\left(\left(\frac{9}{4}\right)^a - 1\right)\ \text{cm}^2$, where $p \in \mathbb{Q}^+, a \in \mathbb{Z}^+$. [3]

Markscheme

attempt to find the sum of the areas using $S_n = \frac{u_1(r^n-1)}{r-1}$ ***(M1)***

sum of areas $\frac{20\left(\left(\frac{9}{4}\right)^{10}-1\right)}{\frac{9}{4}-1}\ \left(= 16\left(\left(\frac{9}{4}\right)^{10} - 1\right)\right)$ ***(A1)***

mean area $= \frac{1}{10}\left(\frac{20\left(\left(\frac{9}{4}\right)^{10}-1\right)}{\frac{9}{4}-1}\right)\ \left(= \frac{1}{10}\left(16\left(\left(\frac{9}{4}\right)^{10} - 1\right)\right)\right)$

$= \frac{16}{10}\left(\left(\frac{9}{4}\right)^{10} - 1\right)\ \left(= \frac{8}{5}\left(\left(\frac{9}{4}\right)^{10} - 1\right)\right)$ ***A1***

$p = \frac{8}{5}, a = 10$

[3 marks]

(b) Find the median area of the ten picture frames, giving your answer in the form $q\left(\frac{9}{4}\right)^4 \text{ cm}^2$, where $q \in \mathbb{Q}^+$. [3]

Markscheme

recognition that median is between 5th and 6th picture frame *(M1)*

median area $= \frac{20\left(\frac{9}{4}\right)^4 + 20\left(\frac{9}{4}\right)^5}{2}$ *(A1)*

$= \frac{20\left(\frac{9}{4}\right)^4\left(1+\frac{9}{4}\right)}{2}$

$= \frac{65}{2}\left(\frac{9}{4}\right)^4$ *A1*

$q = \frac{65}{2}$

[3 marks]

2. [Maximum mark: 16]

Consider the sequence $\{u_n\}$, with $n\text{th}$ term given by u_n. The first three terms are

$u_1 = k - 5,\ u_2 = 3 - 2k$ and $u_3 = 5k + 3$, where $k \in \mathbb{R}$.

(a) Consider the case when $\{u_n\}$ is arithmetic.

(a.i) Find the value of k. [3]

Markscheme

METHOD 1

attempt to equate differences of consecutive terms *(M1)*

$(3 - 2k) - (k - 5) = (5k + 3) - (3 - 2k)$ **OR**
$(k - 5) - (3 - 2k) = (3 - 2k) - (5k + 3)$ *A1*

$8 - 3k = 7k$

$(k =)\ \frac{4}{5}$ *A1*

METHOD 2 (system of equations)

<u>TWO</u> correct equations involving k and d *A1*

$k-5+d=3-2k$ **OR** $3-2k+d=5k+3$ **OR** $k-5+2d=5k+3$ **OR**
$\frac{3}{2}(2(k-5)+2d)=k-5+3-2k+5k+3$ (or equivalent)

valid attempt to solve their system of equations using substitution or elimination ***(M1)***

$(d=5.6)$

$(k=)\ \frac{4}{5}$ ***A1***

METHOD 3 (in terms of k)

$\frac{3}{2}(k-5+5k+3)=k-5+3-2k+5k+3$ (or equivalent) ***A1***

combining like terms ***(M1)***

$9k-3=4k+1$ **OR** $5k=4$ (or equivalent)

$(k=)\ \frac{4}{5}$ ***A1***

METHOD 4 (arithmetic mean)

attempt to find mean of u_1 and u_3 ***(M1)***

$\frac{(k-5)+(5k+3)}{2}=3-2k$ ***A1***

$3k-1=3-2k$

$(k=)\ \frac{4}{5}$ ***A1***

[3 marks]

(a.ii) Hence, or otherwise, find u_3. [2]

Markscheme

substituting their value of k into expression for u_3 ***(A1)***

$(u_3=)5\times\frac{4}{5}+3$

$=7$ ***A1***

[2 marks]

(b) Consider the case where $k=12$.

(b.i) Show that the first three terms of $\{u_n\}$ form a geometric sequence.

[3]

Markscheme

substituting $k = 12$ into $u_1,\ u_2$ or u_3 ***(M1)***

$(u_1 =)\ 7$ **AND** $(u_2 =)\ -21$ **AND** $(u_3 =)\ 63$ ***(A1)***

$(r =) \frac{-21}{7} = \frac{63}{-21}\ (= -3)$ **OR** $(-21)^2 = 7 \times 63$ **OR** $r = -3$ ***R1***

$u_1,\ u_2$ and u_3 are in geometric sequence ***AG***

[3 marks]

(b.ii) Given that $\{u_n\}$ is geometric, state a reason why the sum of an infinite number of terms of this sequence does not exist. [1]

Markscheme

since $|r| \geq 1$ ***R1***

hence not convergent ***AG***

[1 mark]

(c) The sequence, $\{u_n\}$, is geometric for a second value of k.

(c.i) Show that $k^2 - 10k - 24 = 0$. [2]

Markscheme

attempts to find ratios, in terms of k, of consecutive terms and equating ***(M1)***

$\frac{(3-2k)}{(k-5)} = \frac{(5k+3)}{(3-2k)}$ **OR** $(3 - 2k)^2 = (k - 5)(5k + 3)$ (or equivalent)

$9 - 12k + 4k^2 = 5k^2 - 22k - 15$ ***A1***

Note: Award ***A1*** for correct expansion of all brackets leading to given result.

$k^2 - 10k - 24 = 0$ ***AG***

[2 marks]

(c.ii) Find the first three terms of $\{u_n\}$ for this second value of k.

[4]

Markscheme

recognizing need to factorize, complete the square or substitute into quadratic formula ***(M1)***

$(k+2)(k-12)\ (=0)$ **OR** $(k-5)^2-49\ (=0)$ **OR** $k=\frac{10\pm\sqrt{196}}{2}$

$k=-2$ (accept $k=-2$ **and** $k=12$) ***A1***

substituting their value of k (other than $k=12$) to find $u_1,\ u_2$ or u_3 ***(M1)***

$(u_1=)-7$ **AND** $(u_2=)7$ **AND** $(u_3=)-7$ ***A1***

[4 marks]

(c.iii) Hence, write down the value of S_{2m}, the sum of the first $2m$ terms, for this second value of k. [1]

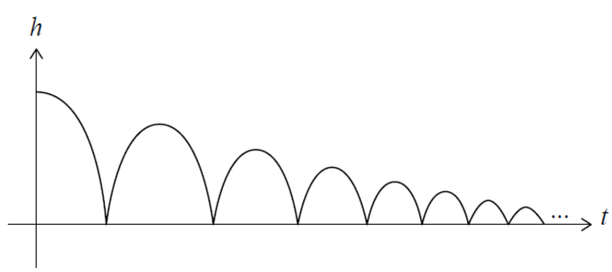
Markscheme

$(S_{2m}=)\ 0$ ***A1***

[1 mark]

3. [Maximum mark: 14]

A tennis ball is dropped from a height. After each bounce the maximum height reached by the ball is $\frac{2}{3}$ of its previous maximum height. This can be seen in the diagram below where h, in metres, is the height of a ball after t seconds.



A box contains tennis balls. Each ball satisfies the condition of rebounding to $\frac{2}{3}$ of their previous maximum height. The tennis balls are numbered Ball $1,\ 2,\ 3\ldots$

Ball 1 is dropped from a height of 10 metres.

(a) Find the maximum height of Ball 1 after the 5^{th} bounce.

[3]

Markscheme

recognising geometric sequence with $r = \frac{2}{3}$ (seen anywhere) ***(M1)***

$10\left(\frac{2}{3}\right)^{6-1}$ **OR** $\frac{20}{3}\left(\frac{2}{3}\right)^{5-1}$ ***(A1)***

$1.31687\ldots$

height after 5^{th} bounce$= 1.32$ (m) (accept $\frac{320}{243}$) ***A1***

[3 marks]

(b) Find the total distance travelled by Ball 1 immediately before the 5^{th} bounce. [3]

Markscheme

recognition of the need to use S_n (seen anywhere) ***(M1)***

$10 + 10 \times \left(\frac{2}{3}\right) + 10 \times \left(\frac{2}{3}\right)^2 + 10 \times \left(\frac{2}{3}\right)^3 + 10 \times \left(\frac{2}{3}\right)^4$ **OR** S_5

recognition to double the height ***(M1)***

$10 + 2 \times \mathrm{S}_4$ **OR** $2 \times \mathrm{S}_5 - 10$ **OR**
$10 + 2 \times 10 \times \left(\frac{2}{3}\right) + 2 \times 10 \times \left(\frac{2}{3}\right)^2 + 2 \times 10 \times \left(\frac{2}{3}\right)^3 + 2 \times 10 \times \left(\frac{2}{3}\right)^4$ (or equivalent)

$42.0987\ldots$

total distance travelled$= 42.1$ ***A1***

[3 marks]

Let δ be the total distance travelled by any of these balls.

(c) A ball is dropped from a height of x metres. Show that $\delta = 5x$ metres. [3]

Markscheme

recognising the need to use S_∞ formula (seen anywhere) ***(M1)***

$2S_\infty + u_1$ **OR** $2S_\infty - x$

$2\left(\frac{\frac{2}{3}x}{1-\frac{2}{3}}\right) + x$ **OR** $2\left(\frac{x}{1-\frac{2}{3}}\right) - x$ **OR** $\frac{x}{1-\frac{2}{3}} + \frac{\frac{2}{3}x}{1-\frac{2}{3}}$ ***A1***

$2(2x) + x$ **OR** $6x - x$ **OR** $3x + 2x$ ***A1*** total distance travelled$= 5x$ ***AG***

[3 marks]

Let δ_1 be the total distance travelled by Ball 1.

(d) Write down the value of δ_1. [1]

Markscheme

$\delta_1 = 50$ (m) ***A1***

[1 mark]

Ball 2 is dropped from a height of 9.56 metres.

Let δ_2 be the total distance travelled by Ball 2, and so on for each ball in the box.

It is given that $\delta_1,\ \delta_2\ ,\ \delta_3\ \ldots$ form an arithmetic sequence.

(e) Determine which tennis ball is the first ball to travel less than 25 metres. [4]

Markscheme

$\delta_2 (= 5 \times 9.56) = 47.8$

$d = 47.8 - 50 (= -2.2)$ **OR** $d = 5 \times -0.44 (= -2.2)$ ***(A1)***

attempt to find n using $n\text{th}$ term of an AP with their d ***(M1)***

$50 + (n-1)(-2.2) < 25$ (accept equations)

$12.3636\ldots$ ***(A1)***

Ball 13 ***A1***

[4 marks]

4. [Maximum mark: 5]

Consider a geometric sequence with first term 1 and common ratio 10.

S_n is the sum of the first n terms of the sequence.

(a) Find an expression for S_n in the form $\frac{a^n-1}{b}$, where $a,\ b \in \mathbb{Z}^+$. [1]

Markscheme

$S_n = \frac{10^n - 1}{9}$ ***A1***

$(a = 10, b = 9)$

[1 mark]

(b) Hence, show that $S_1 + S_2 + S_3 + \ldots + S_n = \frac{10(10^n - 1) - 9n}{81}$. [4]

Markscheme

METHOD 1

$S_1 + S_2 + S_3 + \ldots + S_n$

$= \frac{10-1}{9} + \frac{10^2-1}{9} + \ldots + \frac{10^n-1}{9}$ ***(A1)***

$= \frac{10-1+10^2-1+10^3-1+\ldots+10^n-1}{9}$ OR $\frac{9(10-1+10^2-1+10^3-1+\ldots+10^n-1)}{81}$

attempt to use geometric series formula on powers of 10, and collect -1's together ***M1***

$10 + 10^2 + 10^3 + \ldots + 10^n = \frac{10(10^n-1)}{10-1}$ and $-1-1-1\ldots = -n$ ***A1***

$= \frac{\frac{10(10^n-1)}{10-1} - n}{9}$ OR $\frac{9\left(\frac{10(10^n-1)}{10-1}\right) - 9n}{81}$ ***A1***

Note: Award ***A1*** for any correct intermediate expression.

$= \frac{10(10^n-1)-9n}{81}$ ***AG***

METHOD 2

attempt to create sum using sigma notation with S_n ***M1***

$\sum_{i=1}^{n} \frac{10^i - 1}{9} \quad \left(= \frac{1}{9}\left(\sum_{i=1}^{n} 10^i - \sum_{i=1}^{n} 1\right)\right)$

$\sum_{i=1}^{n} 10^i = \frac{10(10^n-1)}{9}$ ***A1***

$\sum_{i=1}^{n} 1 = n$ ***A1***

$= \frac{1}{9}\left(\frac{10(10^n-1)}{9} - n\right)$ OR $\frac{1}{9}\left(\frac{10(10^n-1)-9n}{9}\right)$ ***A1***

$= \frac{10(10^n-1)-9n}{81}$ ***AG***

[4 marks]

5. [Maximum mark: 16]

Consider the arithmetic sequence $a,\ p,\ q\ldots$, where $a,\ p,\ q \neq 0$.

(a) Show that $2p - q = a$. [2]

Markscheme

attempt to find a difference ***(M1)***

$d = p - a, 2d = q - a, d = q - p$ OR $p = a + d, q = a + 2d, q = p + d$

correct equation ***A1***

$p - a = q - p$ OR $q - a = 2(p - a)$ OR $p = \frac{a+q}{2}$ (or equivalent)

$2p - q = a$ ***AG***

[2 marks]

Consider the geometric sequence $a,\ s,\ t\ldots$, where $a,\ s,\ t \neq 0$.

(b) Show that $s^2 = at$. [2]

Markscheme

attempt to find a ratio ***(M1)***

$r = \frac{s}{a},\ r^2 = \frac{t}{a},\ r = \frac{t}{s}$ OR $s = ar, t = ar^2, t = sr$

correct equation ***A1***

$\left(\frac{s}{a}\right)^2 = \frac{t}{a}$ OR $\frac{s}{a} = \frac{t}{s}$ (or equivalent)

$s^2 = at$ ***AG***

[2 marks]

The first term of both sequences is a.

It is given that $q = t = 1$.

(c) Show that $p > \frac{1}{2}$. [2]

Markscheme

EITHER

$2p - 1 = s^2$ (or equivalent) ***A1***

$\left(s^2 > 0\right) \Rightarrow 2p - 1 > 0$ OR $s = \sqrt{2p-1} \Rightarrow 2p - 1 > 0$ OR $p = \frac{s^2+1}{2}$ (and $s^2 > 0$)
R1

OR

$2p - 1 = a$ and $s^2 = a$ ***A1***

($s^2 > 0$, so) $a > 0 \Rightarrow 2p - 1 > 0$ OR $p\frac{a+1}{2}$ and $a > 0$ ***R1***

$\Rightarrow p > \frac{1}{2}$ ***AG***

Note: Do not award ***A0R1***.

[2 marks]

Consider the case where $a = 9,\ s > 0$ and $q = t = 1$.

(d) Write down the first four terms of the

(d.i) arithmetic sequence; [2]

Markscheme

$9, 5, 1, -3$ ***A1A1***

Note: Award ***A1*** for each of 2nd term and 4th term

[2 marks]

(d.ii) geometric sequence. [2]

Markscheme

$9, 3, 1, \frac{1}{3}$ ***A1A1***

Note: Award ***A1*** for each of 2nd term and 4th term

[2 marks]

The arithmetic and the geometric sequence are used to form a new arithmetic sequence u_n.

The first three terms of u_n are $u_1 = 9 + \ln 9,\ u_2 = 5 + \ln 3$, and $u_3 = 1 + \ln 1$.

(e.i) Find the common difference of the new sequence in terms of $\ln 3$. [3]

Markscheme

attempt to find the difference between two consecutive terms ***(M1)***

$d = u_2 - u_1 = 5 + \ln 3 - 9 - \ln 9$ OR $d = u_3 - u_2 = 1 + \ln 1 - 5 - \ln 3$

$\ln 9 = 2 \ln 3$ OR $\ln 1 = 0$ OR $\ln 3 - \ln 9 = \ln \frac{1}{3} \left(= \ln 3^{-1} = -\ln 3\right)$ (seen anywhere) ***(A1)***

$d = -4 - \ln 3$ ***A1***

[3 marks]

(e.ii) Show that $\sum_{i=1}^{10} = -90 - 25 \ln 3$. [3]

Markscheme

METHOD 1

attempt to substitute first term and their common difference into S_{10} ***(M1)***

$\frac{10}{2}(2(9 + \ln 9) + 9(-4 - \ln 3))$ OR $\frac{10}{2}(2(9 + 2 \ln 3) + 9(-4 - \ln 3))$ (or equivalent) ***A1***

$= 5(-18 - 5 \ln 3)$ (or equivalent in terms of $\ln 3$) ***A1***

$\sum_{i=1}^{10} u_i = -90 - 25 \ln 3$ ***AG***

METHOD 2

$u_{10} = 9 + \ln 9 + 9(-4 - \ln 3)(= -27 + \ln 9 - 9 \ln 3)$

attempt to substitute first term and their u_{10} into S_{10} ***(M1)***

$\frac{10}{2}(2(9+\ln 9)+9(-4-\ln 3))$ OR $\frac{10}{2}(9+\ln 9-27+\ln 9-9\ln 3)$ OR

$\frac{10}{2}(2(9+2\ln 3)+9(-4-\ln 3))$ OR $\frac{10}{2}(9+\ln 9-27-7\ln 3)$ (or equivalent) ***A1***

$=5(-18-5\ln 3)$ (or equivalent in terms of $\ln 3$) ***A1***

$\sum_{i=1}^{10} u_i=-90-25\ln 3$ ***AG***

[3 marks]

6. [Maximum mark: 7]

Darren buys a car for $\$35\,000$. The value of the car decreases by $15\,\%$ in the first year.

(a) Find the value of the car at the end of the first year. [2]

Markscheme

recognition that a $15\,\%$ loss leaves $85\,\%$ OR finding $15\,\%$ and subtracting from original ***(M1)***

$0.85\times 35\,000$ OR $35\,000-0.15\times 35\,000$

$=(\$)29\,750$ ***A1***

Note: Accept $(\$)29\,800$.

[2 marks]

After the first year, the value of the car decreases by $11\,\%$ in each subsequent year.

(b) Find the value of Darren's car 10 years after he buys it, giving your answer to the nearest dollar. [2]

Markscheme

EITHER

$29\,750\times 0.89^9$ ***(A1)***

OR

$\text{N}=9$

$\text{I}\%=-11$

$\text{PV} = \mp 29\,750$ ***(A1)***

THEN

value $(FV) = (\$)10\,423$ ***A1***

Note: For this ***A1*** the answer must be rounded to the nearest dollar. Accept $(\$)10\,441$ from using 3 sf answer from part (a).

[2 marks]

When Darren has owned the car for n complete years, the value of the car is less than 10% of its original value.

(c) Find the least value of n. [3]

Markscheme

METHOD 1

attempt to solve the inequality (or equation) $29\,750 \times 0.89^{n-1} < 3500$ OR table of values ***(M1)***

$19.3643\ldots$ OR $(n = 19 \Rightarrow) 3651.80\ldots$ OR $(n = 20 \Rightarrow) 3250.10\ldots$ ***(A1)***

Note: For candidates using $(\$)\,29\,800,\ n > 19.3787\ldots,\ (n = 19 \Rightarrow) 3657.93\ldots,\ (n = 20 \Rightarrow) 3255.56\ldots$.

$n = 20$ ***A1***

METHOD 2

use of the finance app with $\text{I}\% = -11,\ \text{PV} = \mp 29\,750,\ \text{FV} = \pm 3500$

OR $29\,750 \times 0.89^{N} < 3500$ (condone the use of n or x) ***(M1)***

$(N =) 18.3643\ldots$ ***(A1)***

Note: For candidates using $(\$)\,29\,800,\ N = 18.3787\ldots$

$n = 20$ ***A1***

[3 marks]

7. [Maximum mark: 14]

Consider the arithmetic sequence $u_1, u_2, u_3, \ldots$.

The sum of the first n terms of this sequence is given by $S_n = n^2 + 4n$.

(a.i) Find the sum of the first five terms. [2]

Markscheme

recognition that $n = 5$ ***(M1)***

$S_5 = 45$ ***A1***

[2 marks]

(a.ii) Given that $S_6 = 60$, find u_6. [2]

Markscheme

METHOD 1

recognition that $S_5 + u_6 = S_6$ ***(M1)***

$u_6 = 15$ ***A1***

METHOD 2

recognition that $60 = \frac{6}{2}(S_1 + u_6)$ ***(M1)***

$60 = 3(5 + u_6)$

$u_6 = 15$ ***A1***

METHOD 3

substituting their u_1 and d values into $u_1 + (n-1)d$ ***(M1)***

$u_6 = 15$ ***A1***

[2 marks]

(b) Find u_1. [2]

Markscheme

recognition that $u_1 = S_1$ (may be seen in (a)) OR substituting their u_6 into S_6 ***(M1)***

OR equations for S_5 and S_6 in terms of u_1 and d

$1 + 4$ OR $60 = \frac{6}{2}\left(U_1 + 15\right)$

$u_1 = 5$ ***A1***

[2 marks]

(c) Hence or otherwise, write an expression for u_n in terms of n. [3]

Markscheme

EITHER

valid attempt to find d (may be seen in (a) or (b)) ***(M1)***

$d = 2$ ***(A1)***

OR

valid attempt to find $S_n - S_{n-1}$ ***(M1)***

$n^2 + 4n - \left(n^2 - 2n + 1 + 4n - 4\right)$ ***(A1)***

OR

equating $n^2 + 4n = \frac{n}{2}\left(5 + u_n\right)$ ***(M1)***

$2n + 8 = 5 + u_n$ (or equivalent) ***(A1)***

THEN

$u_n = 5 + 2(n - 1)$ OR $u_n = 2n + 3$ ***A1***

[3 marks]

Consider a geometric sequence, v_n, where $v_2 = u_1$ and $v_4 = u_6$.

(d) Find the possible values of the common ratio, r. [3]

Markscheme

recognition that $v_2 r^2 = v_4$ OR $(v_3)^2 = v_2 \times v_4$ ***(M1)***

$r^2 = 3$ OR $v_3 = (\pm) 5\sqrt{3}$ ***(A1)***

$r = \pm\sqrt{3}$ ***A1***

Note: If no working shown, award ***M1A1A0*** for $\sqrt{3}$.

[3 marks]

(e) Given that $v_{99} < 0$, find v_5. [2]

Markscheme

recognition that r is negative ***(M1)***

$v_5 = -15\sqrt{3} \left(= -\frac{45}{\sqrt{3}}\right)$ ***A1***

[2 marks]

8. [Maximum mark: 15]

Consider the series $\ln x + p \ln x + \frac{1}{3} \ln x + \ldots$, where $x \in \mathbb{R},\ x > 1$ and $p \in \mathbb{R},\ p \neq 0$.

Consider the case where the series is geometric.

(a.i) Show that $p = \pm\frac{1}{\sqrt{3}}$. [2]

Markscheme

EITHER

attempt to use a ratio from consecutive terms ***M1***

$\frac{p\ln x}{\ln x} = \frac{\frac{1}{3}\ln x}{p\ln x}$ OR $\frac{1}{3}\ln x = (\ln x)r^2$ OR $p\ln x = \ln x\left(\frac{1}{3p}\right)$

Note: Candidates may use $\ln x^1 + \ln x^p + \ln x^{\frac{1}{3}}\ldots$ and consider the powers of x in geometric sequence

Award ***M1*** for $\frac{p}{1} = \frac{\frac{1}{3}}{p}$.

OR

$r = p$ and $r^2 = \frac{1}{3}$ ***M1***

THEN

$p^2 = \frac{1}{3}$ OR $r = \pm\frac{1}{\sqrt{3}}$ ***A1***

$p = \pm\frac{1}{\sqrt{3}}$ ***AG***

Note: Award ***M0A0*** for $r^2 = \frac{1}{3}$ or $p^2 = \frac{1}{3}$ with no other working seen.

[2 marks]

(a.ii) Given that $p > 0$ and $S_\infty = 3 + \sqrt{3}$, find the value of x. [3]

Markscheme

$\frac{\ln x}{1-\frac{1}{\sqrt{3}}}$ $\left(= 3 + \sqrt{3}\right)$ ***(A1)***

$\ln x = 3 - \frac{3}{\sqrt{3}} + \sqrt{3} - \frac{\sqrt{3}}{\sqrt{3}}$ OR $\ln x = 3 - \sqrt{3} + \sqrt{3} - 1$ $(\Rightarrow \ln x = 2)$ ***A1***

$x = e^2$ ***A1***

[3 marks]

Now consider the case where the series is arithmetic with common difference d.

(b.i) Show that $p = \frac{2}{3}$. [3]

Markscheme

METHOD 1

attempt to find a difference from consecutive terms or from u_2 ***M1***

correct equation ***A1***

$p \ln x - \ln x = \frac{1}{3} \ln x - p \ln x$ OR $\frac{1}{3} \ln x = \ln x + 2(p \ln x - \ln x)$

Note: Candidates may use $\ln x^1 + \ln x^p + \ln x^{\frac{1}{3}} + \ldots$ and consider the powers of x in arithmetic sequence.

Award ***M1A1*** for $p - 1 = \frac{1}{3} - p$

$2p \ln x = \frac{4}{3} \ln x \ \left(\Rightarrow 2p = \frac{4}{3}\right)$ ***A1***

$p = \frac{2}{3}$ ***AG***

METHOD 2

attempt to use arithmetic mean $u_2 = \frac{u_1 + u_3}{2}$ ***M1***

$p \ln x = \frac{\ln x + \frac{1}{3} \ln x}{2}$ ***A1***

$2p \ln x = \frac{4}{3} \ln x \ \left(\Rightarrow 2p = \frac{4}{3}\right)$ ***A1***

$p = \frac{2}{3}$ ***AG***

METHOD 3

attempt to find difference using u_3 ***M1***

$\frac{1}{3} \ln x = \ln x + 2d \ \left(\Rightarrow d = -\frac{1}{3} \ln x\right)$

$u_2 = \ln x + \frac{1}{2}\left(\frac{1}{3} \ln x - \ln x\right)$ OR $p \ln x - \ln x = -\frac{1}{3} \ln x$ ***A1***

$p \ln x = \frac{2}{3} \ln x$ ***A1***

$p = \frac{2}{3}$ ***AG***

[3 marks]

(b.ii) Write down d in the form $k \ln x$, where $k \in \mathbb{Q}$. [1]

Markscheme

$d = -\frac{1}{3}\ln x$ ***A1***

[1 mark]

(b.iii) The sum of the first n terms of the series is $-3 \ln x$.

Find the value of n. [6]

Markscheme

METHOD 1

$S_n = \frac{n}{2}\left[2 \ln x + (n-1) \times \left(-\frac{1}{3}\ln x\right)\right]$

attempt to substitute into S_n and equate to $-3 \ln x$ ***(M1)***

$\frac{n}{2}\left[2 \ln x + (n-1) \times \left(-\frac{1}{3}\ln x\right)\right] = -3 \ln x$

correct working with S_n (seen anywhere) ***(A1)***

$\frac{n}{2}\left[2 \ln x - \frac{n}{3}\ln x + \frac{1}{3}\ln x\right]$ OR $n \ln x - \frac{n(n-1)}{6}\ln x$ OR $\frac{n}{2}\left(\ln x + \left(\frac{4-n}{3}\right) \ln x\right)$

correct equation without $\ln x$ ***A1***

$\frac{n}{2}\left(\frac{7}{3} - \frac{n}{3}\right) = -3$ OR $n - \frac{n(n-1)}{6} = -3$ or equivalent

Note: Award as above if the series $1 + p + \frac{1}{3} + \ldots$ is considered leading to $\frac{n}{2}\left(\frac{7}{3} - \frac{n}{3}\right) = -3$.

attempt to form a quadratic $= 0$ ***(M1)***

$n^2 - 7n - 18 = 0$

attempt to solve their quadratic ***(M1)***

$(n-9)(n+2) = 0$

$n = 9$ ***A1***

METHOD 2

listing the first 7 terms of the sequence ***(A1)***

$\ln x + \frac{2}{3}\ln x + \frac{1}{3}\ln x + 0 - \frac{1}{3}\ln x - \frac{2}{3}\ln x - \ln x + \ldots$

recognizing first 7 terms sum to 0 ***M1***

8^{th} term is $-\frac{4}{3}\ln x$ ***(A1)***

9^{th} term is $-\frac{5}{3}\ln x$ ***(A1)***

sum of 8^{th} and 9^{th} term $= -3\ln x$ ***(A1)***

$n = 9$ ***A1***

[6 marks]

9. [Maximum mark: 15]

Consider the function $f(x) = a^x$ where $x,\ a \in \mathbb{R}$ and $x > 0,\ a > 1$.

The graph of f contains the point $\left(\frac{2}{3}, 4\right)$.

(a) Show that $a = 8$. [2]

Markscheme

$f\left(\frac{2}{3}\right) = 4$ OR $a^{\frac{2}{3}} = 4$ ***(M1)***

$a = 4^{\frac{3}{2}}$ OR $a = \left(2^2\right)^{\frac{3}{2}}$ OR $a^2 = 64$ OR $\sqrt[3]{a} = 2$ ***A1***

$a = 8$ ***AG***

[2 marks]

(b) Write down an expression for $f^{-1}(x)$. [1]

Markscheme

$f^{-1}(x) = \log_8 x$ ***A1***

Note: Accept $f^{-1}(x) = \log_a x$.

Accept any equivalent expression for f^{-1} e.g. $f^{-1}(x) = \frac{\ln x}{\ln 8}$.

[1 mark]

(c) Find the value of $f^{-1}\left(\sqrt{32}\right)$. [3]

Markscheme

correct substitution *(A1)*

$\log_8 \sqrt{32}$ OR $8^x = 32^{\frac{1}{2}}$

correct working involving log/index law *(A1)*

$\frac{1}{2}\log_8 32$ OR $\frac{5}{2}\log_8 2$ OR $\log_8 2 = \frac{1}{3}$ OR $\log_2 2^{\frac{5}{2}}$ OR $\log_2 8 = 3$ OR $\frac{\ln 2^{\frac{5}{2}}}{\ln 2^3}$ OR $2^{3x} = 2^{\frac{5}{2}}$

$f^{-1}\left(\sqrt{32}\right) = \frac{5}{6}$ ***A1***

[3 marks]

Consider the arithmetic sequence $\log_8 27\ ,\ \log_8 p\ ,\ \log_8 q\ ,\ \log_8 125$, where $p > 1$ and $q > 1$.

(d.i) Show that $27,\ p,\ q$ and 125 are four consecutive terms in a geometric sequence. [4]

Markscheme

METHOD 1

equating a pair of differences *(M1)*

$u_2 - u_1 = u_4 - u_3 (= u_3 - u_2)$

$\log_8 p - \log_8 27 = \log_8 125 - \log_8 q$

$\log_8 125 - \log_8 q = \log_8 q - \log_8 p$

$\log_8\left(\frac{p}{27}\right) = \log_8\left(\frac{125}{q}\right),\ \log_8\left(\frac{125}{q}\right) = \log_8\left(\frac{q}{p}\right)$ ***A1A1***

$\frac{p}{27} = \frac{125}{q}$ and $\frac{125}{q} = \frac{q}{p}$ ***A1***

$27,\ p,\ q$ and 125 are in geometric sequence ***AG***

Note: If candidate assumes the sequence is geometric, award no marks for part (i). If $r = \frac{5}{3}$ has been found, this will be awarded marks in part (ii).

METHOD 2

expressing a pair of consecutive terms, in terms of d ***(M1)***

$p = 8^d \times 27$ and $q = 8^{2d} \times 27$ OR $q = 8^{2d} \times 27$ and $125 = 8^{3d} \times 27$

two correct pairs of consecutive terms, in terms of d ***A1***

$\frac{8^d \times 27}{27} = \frac{8^{2d} \times 27}{8^d \times 27} = \frac{8^{3d} \times 27}{8^{2d} \times 27}$ (must include 3 ratios) ***A1***

all simplify to 8^d ***A1***

$27,\ p,\ q$ and 125 are in geometric sequence ***AG***

[4 marks]

(d.ii) Find the value of p and the value of q. [5]

Markscheme

METHOD 1 (geometric, finding r)

$u_4 = u_1 r^3$ OR $125 = 27(r)^3$ ***(M1)***

$r = \frac{5}{3}$ (seen anywhere) ***A1***

$p = 27r$ OR $\frac{125}{q} = \frac{5}{3}$ ***(M1)***

$p = 45,\ q = 75$ ***A1A1***

METHOD 2 (arithmetic)

$u_4 = u_1 + 3d$ OR $\log_8 125 = \log_8 27 + 3d$ ***(M1)***

$d = \log_8\left(\frac{5}{3}\right)$ (seen anywhere) ***A1***

$\log_8 p = \log_8 27 + \log_8\left(\frac{5}{3}\right)$ OR $\log_8 q = \log_8 27 + 2\log_8\left(\frac{5}{3}\right)$ ***(M1)***

$p = 45,\ q = 75$ ***A1A1***

METHOD 3 (geometric using proportion)

recognizing proportion *(M1)*

$pq = 125 \times 27$ OR $q^2 = 125p$ OR $p^2 = 27q$

two correct proportion equations *A1*

attempt to eliminate either p or q *(M1)*

$q^2 = 125 \times \frac{125 \times 27}{q}$ OR $p^2 = 27 \times \frac{125 \times 27}{p}$

$p = 45,\ q = 75$ *A1A1*

[5 marks]

10. [Maximum mark: 9]

The sum of the first n terms of a geometric sequence is given by $S_n = \sum_{r=1}^{n} \frac{2}{3}\left(\frac{7}{8}\right)^r$.

(a) Find the first term of the sequence, u_1. [2]

Markscheme

$u_1 = S_1 = \frac{2}{3} \times \frac{7}{8}$ *(M1)*

$= \frac{14}{24}\left(= \frac{7}{12} = 0.583333\ldots\right)$ *A1*

[2 marks]

(b) Find S_∞. [3]

Markscheme

$r = \frac{7}{8}(= 0.875)$ *(A1)*

substituting their values for u_1 and r into $S_\infty = \frac{u_1}{1-r}$ *(M1)*

$= \frac{14}{3}(= 4.66666\ldots)$ *A1*

[3 marks]

(c) Find the least value of n such that $S_\infty - S_n < 0.001$. [4]

Markscheme

attempt to substitute their values into the inequality or formula for S_n ***(M1)***

$\frac{14}{3} - \sum_{r=1}^{n} \frac{2}{3}\left(\frac{7}{8}\right)^r < 0.001$ OR $S_n = \frac{\frac{7}{12}\left(1-\left(\frac{7}{8}\right)^n\right)}{\left(1-\frac{7}{8}\right)}$

attempt to solve their inequality using a table, graph or logarithms

(must be exponential) ***(M1)***

Note: Award ***(M0)*** if the candidate attempts to solve $S_\infty - u_n < 0.001$.

correct critical value or at least one correct crossover value ***(A1)***

$63.2675\ldots$ OR $S_\infty - S_{63} = 0.001036\ldots$ OR $S_\infty - S_{64} = 0.000906\ldots$

OR $S_\infty - S_{63} - 0.001 = 0.0000363683\ldots$ OR
$S_\infty - S_{64} - 0.001 = 0.0000931777\ldots$

least value is $n = 64$ ***A1***

[4 marks]

11. [Maximum mark: 6]
Mia baked a very large apple pie that she cuts into slices to share with her friends. The smallest slice is cut first. The volume of each successive slice of pie forms a geometric sequence.

The second smallest slice has a volume of $30\,\text{cm}^3$. The fifth smallest slice has a volume of $240\,\text{cm}^3$.

(a) Find the common ratio of the sequence. [2]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure. It appeared in a paper that permitted the use of a calculator, and so might not be suitable for all forms of practice.

$u_1 r = 30$ **and** $u_1 r^4 = 240,$ ***(M1)***

Note: Award ***(M1)*** for both the given terms expressed in the formula for u_n.

OR

$30r^3 = 240 \quad (r^3 = 8)$ ***(M1)***

Note: Award ***(M1)*** for a correct equation seen.

$(r =)\ 2$ ***(A1)*** ***(C2)***

[2 marks]

(b) Find the volume of the smallest slice of pie. [2]

Markscheme

$u_1 \times 2 = 30$ **OR** $u_1 \times 2^4 = 240$ ***(M1)***

Note: Award ***(M1)*** for their correct substitution in geometric sequence formula.

$(u_1 =)\ 15$ ***(A1)*(ft)** ***(C2)***

Note: Follow through from part (a).

[2 marks]

(c) The apple pie has a volume of $61\,425\,\mathrm{cm}^3$.

Find the total number of slices Mia can cut from this pie. [2]

Markscheme

$\frac{15(2^n-1)}{2-1} = 61425$ ***(M1)***

Note: Award ***(M1)*** for correctly substituted geometric series formula equated to 61425.

$(n =)\ 12$ (slices) ***(A1)*(ft)** ***(C2)***

Note: Follow through from parts (a) and (b).

[2 marks]